# A representation theorem for cardinal algebras

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# Cardinal algebras

A **cardinal algebra** (Tarski 1949) $A = (A, +, 0, \sum)$ consists of

- a commutative monoid $(A, +, 0)$;
- an infinitary operation $\sum : A^{\mathbb{N}} \to A$; obeying the axioms
- $a_0 + \sum_{i<\infty} a_{i+1} =$ $\sum_i a_i := \sum(a_0, a_1, \dots)$
- $\sum_{i<\infty}(a_i + b_i) =$ $\sum_i a_i + \sum_i b_i$
- (**refinement**) $\forall a + b = \sum_{i<\infty} c_i,$ $\exists (a_i)_i, (b_i)_i$ s.t.

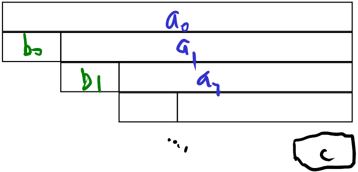
| | $c_0$ | $c_1$ | $c_2$ | $c_3$ | $\cdots$ |
|---|---|---|---|---|---|
| $a$ | $a_0$ | $a_1$ | $a_2$ | $\dots$ | |
| $b$ | $b_0$ | $b_1$ | $b_2$ | $\dots$ | |

$a = \sum_i a_i$

$b = \sum_i b_i$

$a_i + b_i = c_i \quad \forall i$

- (**remainder**) $\forall (a_i = b_i + a_{i+1})_{i \in \mathbb{N}},$ $\exists c$ s.t. $a_i = c + \sum_j b_{i+j}$ $\forall i.$



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$$E \sim_B F \;:\Longleftrightarrow\; \exists \text{ Borel } f : X \to Y \text{ inducing } X/E \cong Y/F$$

  Then $\{\text{all CBERS}\}/{\sim_B}$ is a cardinal algebra under $\bigsqcup$.

# Properties of cardinal algebras

- (Tarski) All commutativity + associativity laws hold, e.g.,

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- If binary meets exist, then so do binary joins, and

$$a \wedge \bigvee_{i<\infty} b_i = \bigvee_{i<\infty}(a \wedge b_i),$$
$$a + (b \wedge c) = (a + b) \wedge (a + c),$$
$$a + (b \vee c) = (a + b) \vee (a + c).$$

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  This extends to a $\sigma$-additive action

  $$\begin{aligned} N \times A &\longrightarrow A \\ (r, a) &\longmapsto r \cdot a \end{aligned}$$

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$(1-\varepsilon)a \leq b \quad \forall \varepsilon > 0,$

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Moral: all "natural" properties of $[0, \infty]$ seem to hold in all cardinal algebras.

# Regular positive $\sigma$-DCPO-monoids

A **universal Horn axiom** (in $\mathcal{L}_{\infty\infty}$) is one of the form

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Regard a card alg $A$ as a $(+, 0, \leq, \bigvee^{\uparrow})$-structure, obeying axioms:

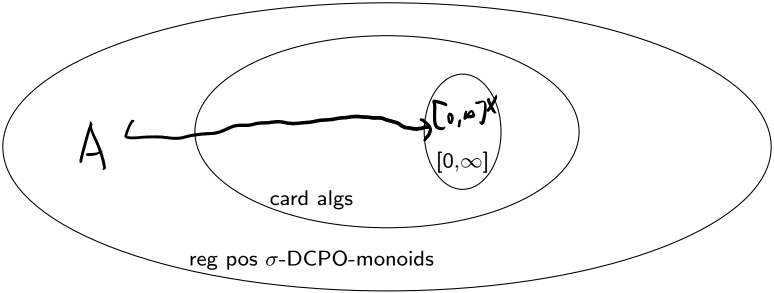
- $(A, \leq, \bigvee^{\uparrow})$ is a $\sigma$-**DCPO**: poset w/ incr. ctbl. joins
- $(A, +, 0)$ is a (comm) **monoid**, s.t. $+$ is monotone & distributes over $\bigvee^{\uparrow}$
- every $a \in A$ is **positive**: $0 \leq a$
- $A$ is **regular**: $\forall n \, (na \leq (n+1)b) \implies a \leq b.$

A general model is a **regular positive $\sigma$-DCPO-monoid**.

# Embedding $\sigma$-DCPO-monoids

**Theorem (C. 2021+)**

*Every reg positive $\sigma$-DCPO-monoid $A$ embeds into some $[0,\infty]^X$.*



# Embedding $\sigma$-DCPO-monoids

$f : X \to [0, \infty]$

$[0,\infty]^2 \to [0,\infty]$ $\quad \mathcal{B}(X) \leftarrow \mathcal{B}([0,\infty])$

## Theorem (C. 2021+)

*Every reg positive $\sigma$-DCPO-monoid $A$ embeds into some $[0, \infty]^X$.*

- *When $A$ is ctbly pres.,* $X$ can be std Borel & → can use Borel maps.
- *In general,* $X$ is a Borel locale.

(Bool $\sigma$-alg, regarded as a formal "Borel space")

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is a ctbly directed colimit of structs which

## Corollary

*The axioms of regular positive $\sigma$-DCPO-monoids axiomatize the (~~countable~~) universal (~~Horn~~) theory of cardinal algebras (~~or $[0, \infty]$~~).*

# With meets and joins

A **regular positive** $\sigma$**-frame-monoid** $(A, +, 0, \leq, \bigvee, \wedge, \infty)$ is:

- a partially ordered comm monoid $(A, +, 0, \leq)$, with $+$ monotone;
- with least element 0 (**positive**) and greatest element $\infty$;
- with ctbl joins and finite meets s.t. $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$;
- $+$ distributes over finite meets and nonempty countable joins;
- $\forall n \, (n \cdot a \leq (n+1) \cdot b) \implies a \leq b$ (**regular**).

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## Proposition (C. 2021+)

*Every reg positive $\sigma$-frame-monoid $A$ embeds into some $[0, \infty]^X$.*

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Previous thm becomes equiv to: every reg pos $\sigma$-DCPO-monoid embeds into a reg pos $\sigma$-frame-monoid.

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  $\inf(b/a) \inf(c/b) \leq \inf(c/a) \implies$ "$[0, \infty]$-enriched poset"

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- Hahn–Banach: every Banach sp admits enough homoms to $\mathbb{R}$.
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Many such thms can be derived constructively from BPIT:

- (In ZF) For a dist lat $A$, $A \hookrightarrow \langle A \rangle_{\mathrm{Bool}} :=$ free Bool alg over $A$.
- So $A \hookrightarrow \langle A \rangle_{\mathrm{Bool}} \underset{\mathrm{BPIT}}{\hookrightarrow} 2^{\mathrm{Hom}_{\mathrm{Bool}}(\langle A \rangle_{\mathrm{Bool}},2)} \cong 2^{\mathrm{Hom}_{\mathrm{DistLat}}(A,2)}$.

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- (Isbell 1972?) Every $\sigma$-**frame** $A$, i.e., poset w/ $\wedge$ and ctbl $\bigvee$ over which $\wedge$ distributes, embeds into $\langle A \rangle_{\sigma\mathrm{Bool}}$.

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- Not every **Boolean $\sigma$-algebra** (ctbly complete Bool alg) admits enough homoms to 2 (e.g., $\mathrm{MALG}([0,1],\lambda)$).
- (Loomis–Sikorski 1940s) For every *ctbly presented* Bool $\sigma$-alg $A$,
$$A \cong \mathcal{B}(\mathrm{Hom}(A,2)) \subseteq 2^{\mathrm{Hom}(A,2)}$$
with image consisting of the $\underbrace{\text{Borel maps on } \mathrm{Hom}(A,2) \subseteq 2^A}_{\mathcal{B}(\mathrm{Hom}(A,2))}$, which is a std Borel space.
- (Isbell 1972?) Every $\sigma$-**frame** $A$, i.e., poset w/ $\wedge$ and ctbl $\bigvee$ over which $\wedge$ distributes, embeds into $\langle A\rangle_{\sigma\mathrm{Bool}}$.
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A **Borel locale** $X$ is an arbitrary Boolean $\sigma$-algebra $\mathcal{B}(X)$.
A **Borel map** $X \to [0,\infty]$ is a Bool $\sigma$-homom $\mathcal{B}([0,\infty]) \to \mathcal{B}(X)$.

# Embedding $\sigma$-frame-monoids: proof sketch

**Proposition (C. 2021+)**

*Every reg positive $\sigma$-frame-monoid $A$ embeds into some $[0, \infty]^X$.*

- *When $A$ is ctbly pres., $[0, \infty]^X =$ Borel maps on std Borel $X$.*
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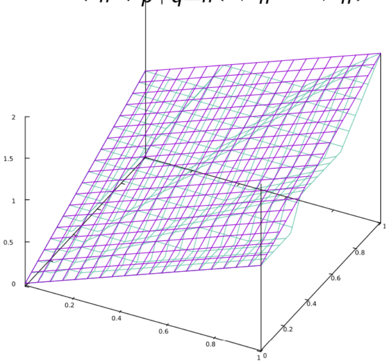
**Lemma (ess. classical)** "$a + b = \bigvee_n \bigvee_{p+q=n} (a/\frac{p}{n} \wedge b/\frac{q}{n})$".

**Sketch.** In $[0, \infty]$, $n = 4$:

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$\{X \to 2\} \cong \{\mathbb{N}^+\text{-eqvar. } X \to 2^{\mathbb{N}^+}\} \hookrightarrow \{\mathbb{N}^+\text{-eqv. } X \to [0,\infty]\}$. $\square$

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**Main Lemma 1** *For a regular positive PO-monoid $A$, its free reg pos completion under meets over which $+$ distributes is presented by*

$$\inf\{q/p \mid pa_n \le qb\} \le 1 \implies \bigwedge_n a_n \le b,$$
$$na_1 \wedge \cdots \wedge na_n + b \le a_1 + \cdots + a_n + b.$$

*Moreover, if $A$ had directed joins over $+$ distributes, then $+$ with a* finite *meet still distributes over these directed joins.*

**Main Lemma 2** *For a reg pos PO-monoid $A$ with $\wedge$, its free reg pos completion under joins over which $+, \wedge$ distribute is presented by*

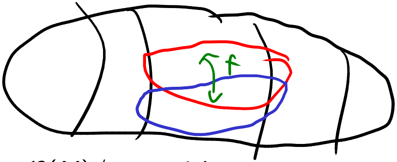
$$\sup\{p/q \mid pa \le qb_n\} \ge 1 \implies a \le \bigvee_n b_n,$$
$$a \wedge b + c \le a + b \implies c \le a \vee b.$$

# Application: Nadkarni's theorem


Fix a CBER $(X, E)$. For Borel $A, B \subseteq X$,

$$A \sim_E B \;:\Longleftrightarrow\; \exists \text{ Borel } f : A \cong B \text{ w/ graph}(f) \subseteq E.$$

$\exists$ Borel $f : X \hookrightarrow Y$ w/ graph$(f) \subseteq E$




Then $\mathcal{K}(E) := \mathcal{B}(X)/\sim_E$, with

$$\sum_i [B_i]_{\sim_E} := [\bigsqcup_i B_i]_{\sim_E} \quad \text{if } B_i \text{ p.w. disjoint}$$

is a cardinal algebra (Tarski), assuming $B_i$ can be disjointified.

If not:

- replace $(X, E)$ with $(X \times \mathbb{N}, E \times \mathbb{I}_{\mathbb{N}})$, or
- replace $\mathcal{B}(X) = \mathcal{B}(X, 2)$ with $\mathcal{B}(X, \overline{\mathbb{N}})$

Note $\mathrm{Hom}(\mathcal{K}(E), [0, \infty]) \cong \{E\text{-invariant measures on } X\} =: \mathrm{INV}^*_E(X)$.

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$X \times \{0\} \not\preceq_{E \times I_{\mathbb{N}}} X \times \mathbb{N}$

**Theorem (Nadkarni 1990, Becker–Kechris 1996)**

$\mathcal{K}(E) \hookrightarrow [0, \infty]^{\mathrm{INV}_E^*(X)}$.

If $A \not\preceq_E B \Rightarrow \exists \mu$ s.t. $\mu(A) > \mu(B)$

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$$A := \left\langle \mathcal{O}(X) \text{ as } \sigma\text{-DCPO} \;\middle|\; \begin{array}{c} \varnothing = 0, \\ U + V = U \cap V + U \cup V, \\ \gamma \cdot U = U \quad \forall \gamma \in \Gamma \end{array} \right\rangle_{\mathsf{RegPos}\sigma\mathsf{DCPOMon}} .$$

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A map $\mu : \mathcal{O}(X) \to [0, \infty]$ preserving $\leq, \bigvee^{\uparrow}$ and satisfying

$$\mu(\varnothing) = 0, \qquad \mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$$

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So $\mathrm{INV}_E(X) \twoheadrightarrow \mathrm{Hom}(A, [0, \infty])$; and $A$ is ctbly presented.
Given $U \not\lesssim V \in \mathcal{K}(E)$, make $U, V$ open; then $U \not\lesssim V \in A$. □

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- For a CBER $E$, the cardinal algebra $\mathcal{K}(E)$ in fact has $\wedge$; homomorphisms preserving these (and $\vee$) correspond to ergodic measures. Showing enough homoms preserving these should be related to ergodic decomposition.